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THE BOUNDARY-LAYER METHOD IN THE FRACTURE MECHANICS OF COMPOSITES OF PERIODIC STRUCTURE*

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The problem of a rectilinear crack in a composite material of doubly - periodic structure is considered. It is assumed that the dimensions of the crack are considerably greater than the cell of material periodicity. A boundary-layer method based on the use of the asymptotic method of averaging periodic structures, taking additional solutions of boundary-layer type /1/ into account to allow the edge effect that occurs near the boundary of the crack outline to be considered, is proposed for analysing the stress field in the neighbourhood of a macrocrack.

Analysis of the stress field in highly inhomogeneous (composite) materials with an idealized smooth macrocrack is usually performed by replacing the inhomogeneous composite medium by a certain homogeneous anisotropic medium that is equivalent to the composite material with respect to the average reaction. Such an approach enables the computation of the average stress field in the composite with a macrocrack to be reduced to solving elasticity theory problems for an anisotropic homogeneous material with a mathematical slit. If the material has a periodic structure (as is true of many composites), the average (effective) characteristics of the equivalent should be determined by the method of averaging periodic structures /1-3/ which yields an asymptotically correct approximation to the exact solution of the problem for the initial inhomogeneous medium. The averaging method here allows the local structure of the fields being investigated to be determined with a high degree of accuracy. This approach was used in /4/ to analyse the stress field near a macrocrack in laminar composites of periodic structure. In a number of cases formulas were obtained for

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the stress intensity factors that express them in terms of the characeristics of the individual composite components and parameters which fix the crack location in the laminar material.

A more rigorous approach to the estimation of the state of stress of a composite material with a crack is proposed.

1. We will confine ourselves to considering a plane problem of elasticity theory for a periodically inhomogeneous (composite) medium with a rectilinear macrocrack whose dimensions are considerably greater than the dimensions of the periodicity cell. We shall also assume that the elastic medium has a doubly-periodic inhomogeneity in the plate x_1, x_2 and the edges of the tunnel crack are parallel to the boundary of the periodicity cell (Fig.1); ε is a dimensionless small parameter that is the ratio between the composite cell dimension and the characteristic body dimension. Within the framework of such a scheme, for example, it is possible to consider a fibrous unidirectional composite material with a tunnel crack whose plane is parallel to the fibre, or a laminar composite with a plane crack located perpendicular (parallel) to the material layers.

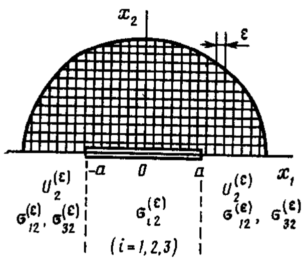

Fig.1

Let the rectilinear macrocrack pass along the periodicity cell boundary of the unbounded domain of the composite material and let a given system of selfequilibrated normal and tangential loads act on its edges. The asymptotic solution of the equations of elasticity theory in a periodically inhomogeneous half-plane $x_2 > 0$ $(x_2 < 0)$ under mixed conditions on the boundary $x_2 = 0$ must be constructed to determine the state of stress and strain in the neighbourhood of such a crack.

The boundary conditions for $x_2 = 0$ correspond to specifying stress $\sigma_{i2}^{(\varepsilon)}(x_1, \pm 0)$ $(i = 1, 2, 3)$ on the section $|x_1| < a$ and mixed conditions on the displacements and stresses for $|x_1| > a$ (Fig.1).

In order to satisfy mixed boundary conditions on the boundary $x_2 = 0$ of a periodically inhomogeneous half-plane $x_2 > 0$ within the framework of the asymptotic method, we will construct the solutions of three auxiliary plane problems for the domain $x_2 > 0$.

2. In the first problem we will seek the solution of the equations

$$\partial \sigma_{i\alpha}^{(\varepsilon)}(x_1, x_2)/\partial x_\alpha = 0 \tag{2.1}$$

$$\sigma_{i\alpha}^{(\varepsilon)}(x_1, x_2) = c_{i\alpha k\beta}(y_1, y_2)\, \partial u_k^{(\varepsilon)}(x_1, x_2)/\partial x_\beta \tag{2.2}$$

$(c_{i\alpha k\beta}(y_1, y_2)$ are singly-periodic functions in the variables $y_\alpha = x_\alpha/\varepsilon$; $i, k = 1, 2, 3$; $\alpha, \beta = 1, 2)$ in the domain $x_2 > 0$ for the following conditions on the boundary:

$$\sigma_{i2}^{(\varepsilon)}(x_1, 0) = p_i(x_1) \tag{2.3}$$

where $p_i(x_1)$ are given functions of the "slow" variable x_1.

Let us postulate the expansion /1-3/

$$u_k^{(\varepsilon)} = u_k^{(0)}(x) + \varepsilon u_k^{(1)}(x, y) + \varepsilon^2 u_k^{(2)}(x, y) + \ldots, (x=(x_1, x_2),\ y=(y_1, y_2)) \tag{2.4}$$

and on the basis of relationships (2.2) and (2.4) we find

$$\sigma_{i\alpha}^{(\varepsilon)} = \sigma_{i\alpha}^{(0)}(x, y) + \varepsilon \sigma_{i\alpha}^{(1)}(x, y) + \ldots \tag{2.5}$$

$$\sigma_{i\alpha}^{(0)}(x, y) = c_{i\alpha k\beta}(y)\left(\partial u_k^{(0)}(x)/\partial x_\beta + \partial u_k^{(1)}(x, y)/\partial y_\beta\right)$$

$$\sigma_{i\alpha}^{(1)}(x, y) = c_{i\alpha k\beta}(y)\left(\partial u_k^{(1)}(x, y)/\partial x_\beta + \partial u_k^{(2)}(x, y)/\partial y_\beta\right)$$

As a result of substituting relationship (2.5) into (2.1) and conditions (2.3) we obtain the following boundary-value problems:

$$\frac{\partial}{\partial y_\alpha}\left[c_{i\alpha k\beta}(y)\frac{\partial u_k^{(1)}(x, y)}{\partial y_\beta}\right] = -\frac{\partial c_{i\alpha k\beta}(y)}{\partial y_\alpha}\frac{\partial u_k^{(0)}(x)}{\partial x_\beta} \tag{2.6}$$

$$c_{i2k\beta}(y)\big|_{y_2=0}\frac{\partial u_k^{(1)}(x, y)}{\partial y_\beta}\bigg|_{y_2=0} = -c_{i2k\beta}(y)\big|_{y_2=0}\frac{\partial u_k^{(0)}(x)}{\partial x_\beta}\bigg|_{x_2=0} + p_i(x_1) \tag{2.7}$$

$$\frac{\partial}{\partial y_\alpha}\left[c_{i\alpha k\beta}(y)\frac{\partial u_k^{(2)}(x, y)}{\partial y_\beta}\right] = -\frac{\partial c_{i\alpha k\beta}(y)}{\partial y_\alpha}\frac{\partial u_k^{(1)}(x, y)}{\partial x_\beta} -$$

$$c_{i\alpha k\beta}(y)\frac{\partial^2 u_k^{(1)}(x, y)}{\partial y_\alpha \partial x_\beta} - c_{i\alpha k\beta}(y)\frac{\partial^2 u_k^{(0)}(x)}{\partial x_\alpha \partial x_\beta} - c_{i\alpha k\beta}(y)\frac{\partial^2 u_k^{(1)}(x, y)}{\partial x_\alpha \partial y_\beta} \tag{2.8}$$

$$c_{i2k\beta}(y)|_{y_2=0}\frac{\partial u_k^{(2)}(x,y)}{\partial y_\beta}\Big|_{y_2=0}+c_{i2k\beta}(y)|_{y_2=0}\frac{\partial u_k^{(1)}(x,y)}{\partial x_\beta}\Big|_{x_2=0}=0 \tag{2.9}$$

We will represent the solution of (2.6) under the conditions (2.7) in the form of the sum of two components

$$u_k^{(1)}(x,y)=u_k^{(1,1)}(x,y)+u_k^{(1,2)}(x,y)$$

where $u_k^{(1,1)}(x,y)$ are singly-periodic functions in the variables y_1, y_2 that satisfy (2.6), while $u_k^{(1,2)}(x,y)$ are functions periodic only in the variable y_1 and determined from the solution of the following problem:

$$\frac{\partial}{\partial y_\alpha}\left[c_{i\alpha k\beta}(y)\frac{\partial u_k^{(1,2)}(x,y)}{\partial y_\beta}\right]=0 \tag{2.10}$$

$$c_{i2k\beta}(y)|_{y_2=0}\frac{\partial u_k^{(1,2)}(x,y)}{\partial y_\beta}\Big|_{y_2=0}=-c_{i2k\beta}(y)|_{y_2=0}\frac{\partial u_k^{(1,1)}(x,y)}{\partial y_\beta}\Big|_{y_2=0}- \tag{2.11}$$

$$c_{i2k\beta}(y)|_{y_2=0}\frac{\partial u_k^{(0)}(x)}{\partial x_\beta}\Big|_{x_2=0}+p_i(x_1);\quad u_k^{(1,2)}(x,y)\to 0 \quad \text{as} \quad y_2\to\infty$$

If we seek $u_n^{(1,1)}(x,y)$ in the form

$$u_n^{(1,1)}(x,y)=N_{nk\beta}(y)\,\partial u_k^{(0)}(x)/\partial x_\beta \tag{2.12}$$

where $N_{nk\beta}(y)$ are singly-periodic functions determinable from the solution of the local problem in the periodicity cell

$$\frac{\partial}{\partial y_\alpha}\left[c_{i\alpha n\gamma}(y)\frac{\partial N_{nk\beta}(y)}{\partial y_\gamma}\right]=-\frac{\partial c_{i\alpha k\beta}(y)}{\partial y_\alpha} \tag{2.13}$$

then it can be shown that the condition for problem (2.10) and (2.11) to be solvable results in the equality

$$C^*_{i2k\beta}\partial u_k^{(0)}(x)/\partial x_\beta|_{x_2=0}=p_i(x_1) \tag{2.14}$$

$$C^*_{i2k\beta}=\int_0^1 C_{i2k\beta}(y_1,0)\,dy_1 \tag{2.15}$$

$$C_{i\alpha k\beta}(y)=c_{i\alpha k\beta}(y)+c_{i\alpha n\gamma}(y)\,\partial N_{nk\beta}(y)/\partial y_\gamma$$

Now taking account of (2.12) and (2.14), we write the first condition of (2.11) in the form

$$c_{i2k\beta}(y)|_{y_2=0}\frac{\partial u_k^{(1,2)}(x,y)}{\partial y_\beta}\Big|_{y_2=0}=[C^*_{i2k\beta}-C_{i2k\beta}(y)|_{y_2=0}]\frac{\partial u_k^{(0)}(x)}{\partial x_\beta}\Big|_{x_2=0} \tag{2.16}$$

The solution of (2.10) can be found from the formula

$$u_n^{(1,2)}(x,y)=N^{(1)}_{nk\beta}(y)\,\partial u_k^{(0)}(x)/\partial x_\beta \tag{2.17}$$

where the functions $N^{(1)}_{nk\beta}(y)$ are determined from the solution of the following boundary-value problem:

$$\frac{\partial}{\partial y_\alpha}\left[c_{i\alpha n\gamma}(y)\frac{\partial N^{(1)}_{nk\beta}(y)}{\partial y_\gamma}\right]=0 \tag{2.18}$$

$$c_{i2n\gamma}(y)|_{y_2=0}\frac{\partial N^{(1)}_{nk\beta}(y)}{\partial y_\gamma}\Big|_{y_2=0}=C^*_{i2k\beta}-C_{i2k\beta}(y)|_{y_2=0}$$

$$N^{(1)}_{nk\beta}(y)\to 0 \quad \text{as} \quad y_2\to\infty$$

where the functions $N^{(1)}_{nk\beta}(y)$ are singly-periodic in y_1.

We will seek the functions $u_k^{(2)}(x,y)$ satisfying the relationships (2.8) and (2.9) also in the form of the sum

$$u_k^{(2)}(x,y)=u_k^{(2,1)}(x,y)+u_k^{(2,2)}(x,y)$$

where $u_k^{(2,1)}(x, y)$ are singly-periodic functions in the variables y_1, y_2, and $u_k^{(2,2)}(x, y)$ are functions periodic in y_1 only which tend to zero as $y_2 \to \infty$. Omitting the awkward equations for the terms $u_k^{(2,1)}(x, y)$ and $u_k^{(2,2)}(x, y)$, we note that the average equations

$$\langle C_{i\alpha k\beta}\rangle \partial^2 u_k^{(0)}(x)/\partial x_\alpha \partial x_\beta = 0 \tag{2.19}$$

and the formulas for the effective characteristics of a homogeneous medium

$$\langle C_{i\alpha k\beta}\rangle = \int_0^1\int_0^1 \left[c_{i\alpha k\beta}(y) + c_{i\alpha n\gamma}(y)\frac{\partial N_{nk\beta}}{\partial y_\gamma}\right] dy_1\, dy_2 \tag{2.20}$$

follow from the solvability conditions for the problem in $u_k^{(2,1)}(x, y)$.

Therefore, within the framework of the zero-th approximation /2/, the displacements in a periodically inhomogeneous half-plane with a given load on the boundary $x_2 = 0$ are determined from the formulas

$$u_n^{(\varepsilon)} \approx u_n^{(0)}(x) + \varepsilon\,[N_{nk\beta}(y) + N_{nk\beta}^{(1)}(y)]\,\partial u_k^{(0)}(x)/\partial x_\beta \tag{2.21}$$

where the vector $u_k^{(0)}(x)$ is the solution of the average problem (2.19) for the half-plane $x_2 > 0$ satisfying condition (2.14).

The expressions for the microstresses in a cell of the composite material here have the form

$$\sigma_{i\alpha}^{(\varepsilon)} \approx \sigma_{i\alpha}^{(0)}(x, y) = \left[c_{i\alpha k\beta}(y) + c_{i\alpha n\gamma}(y)\left(\frac{\partial N_{nk\beta}(y)}{\partial y_\gamma} + \frac{\partial N_{nk\beta}^{(1)}(y)}{\partial y_\gamma}\right)\right]\frac{\partial u_k^{(0)}(x)}{\partial x_\beta} \tag{2.22}$$

3. Let us examine the second auxiliary problem for a periodically inhomogeneous half-plane with given displacements on its boundary

$$u_i^{(\varepsilon)}\big|_{x_2=0} = v_i(x_1) \tag{3.1}$$

If expansion (2.4) is used, it follows from condition (3.1) that

$$u_i^{(0)}(x)\big|_{x_2=0} = v_i(x_1) \tag{3.2}$$

where the functions $u_i^{(1)}(x, y)$ and $u_i^{(2)}(x, y)$ should satisfy (2.6) and (2.8) and the conditions

$$u_i^{(1)}(x, y)\big|_{x_2=0} = 0, \quad u_i^{(2)}(x, y)\big|_{x_2=0} = 0 \tag{3.3}$$

As in the preceding problem, we seek $u_k^{(1)}(x, y)$ in the form of a sum of two components $u_k^{(1,1)}(x, y)$ and $u_k^{(1,2)}(x, y)$. An asymptotic analysis of this problem is performed in /1/ and it is shown that the functions mentioned can be represented in the form

$$u_n^{(1,1)}(x, y) = N_{nk\beta}(y)\frac{\partial u_k^{(0)}(x)}{\partial x_\beta}, \quad u_n^{(1,2)}(x, y) = N_{nk\beta}^{(2)}(y)\frac{\partial u_k^{(0)}(x)}{\partial x_\beta}$$

where $N_{nk\beta}(y)$ are single-periodic functions in the variables y_1, y_2 satisfying Eqs.(2.13) while $N_{nk\beta}^{(2)}(y)$ are singly-periodic solutions in y_1 of the following problem for the half-plane $y_2 > 0$

$$\frac{\partial}{\partial y_\alpha}\left[c_{i\alpha n\gamma}(y)\frac{\partial N_{nk\beta}^{(2)}(y)}{\partial y_\gamma}\right] = 0 \tag{3.4}$$

$$N_{nk\beta}^{(2)}(y)\big|_{y_2=0} = -N_{nk\beta}(y)\big|_{y_2=0} + h_{nk\beta}^{(2)}, \quad N_{nk\beta}^{(2)}(y) \to 0 \quad \text{as} \quad y_2 \to \infty \tag{3.5}$$

The constants $h_{nk\beta}^{(2)}$, in the first condition of (3.5) are determined uniquely from the condition for problem (3.4) and (3.5) to be solvable (see /1/).

In this case the solutions of (2.8) under zero conditions on the half-plane boundary can also be determined in the form of a sum of two components, the first of which is a solution periodic in y_1, y_2 while the second is periodic in y_1 only and tends to zero as $y_2 \to \infty$. The global (average) Eqs.(2.19) here follow from the condition for the local problem to be solvable for the periodic component in y_1, y_2 of the total solution.

Thus, within the framework of the zero-th approximation we have the following expression for the displacements in a periodically inhomogeneous half-plane

$$u_n^{(\varepsilon)} \approx u_n^{(0)}(x) + \varepsilon\,[N_{nk\beta}(y) + N_{nk\beta}^{(2)}(y)]\,\partial u_k^{(0)}(x)/\partial x_\beta \tag{3.6}$$

where $u_k^{(0)}(x)$ are determined from the solution of the average problem, Eqs.(2.19) and conditions (3.2), while the formulas

$$\sigma_{i\alpha}^{(\varepsilon)} \approx \sigma_{i\alpha}^{(0)}(x, y) = \left[c_{i\alpha k\beta}(y) + c_{i\alpha n\gamma}(y)\left(\frac{\partial N_{nk\beta}(y)}{\partial y_\gamma} + \frac{\partial N_{nk\beta}^{(2)}(y)}{\partial y_\gamma}\right)\right]\frac{\partial u_k^{(0)}(x)}{\partial x_\beta} \tag{3.7}$$

hold for the microstress components in the composite cell.

4. If mixed boundary conditions are given on the boundary $x_2 = 0$ of the inhomogeneous half-plane $x_2 > 0$ in the form, for example,

$$\sigma_{i2}^{(\varepsilon)}|_{x_2=0} = p_i(x_1) \quad (i = 1, 3), \quad u_2^{(\varepsilon)}|_{x_2=0} = v_2(x_1) \tag{4.1}$$

it can be shown that the asymptotic solution of the problem has the form (2.21) and (2.22) as before, but the functions $N_{nk\beta}^{(3)}(y)$, that are determined from the solution of the following problem in the domain $y_2 > 0$

$$\frac{\partial}{\partial y_\alpha}\left[c_{i\alpha n\gamma}(y)\frac{\partial N_{nk\beta}^{(3)}(y)}{\partial y_\gamma}\right] = 0 \quad (i, k, n = 1, 2, 3) \tag{4.2}$$

$$c_{i2n\gamma}(y)|_{y_2=0}\left.\frac{\partial N_{nk\beta}^{(3)}(y)}{\partial y_\gamma}\right|_{y_2=0} = C_{i2k\beta}^{*} - C_{i2k\beta}(y)|_{y_2=0} \quad (i = 1, 3) \tag{4.3}$$

$$N_{2k\beta}^{(3)}(y)|_{y_2=0} = -N_{2k\beta}(y)|_{y_2=0} + h_{2k\beta}^{(3)}; \quad N_{nk\beta}^{(3)}(y) \to 0 \quad \text{as} \quad y_2 \to \infty$$

should be substituted in place of the functions $N_{nk\beta}^{(1)}(y)$ in these formulas, where the functions $N_{nk\beta}^{(3)}(y)$ are singly-periodic in y_1.

In this case the vector $u_k^{(0)}(x)$ is the solution of (2.19) under the conditions

$$C_{i2k\beta}^{*}\partial u_k^{(0)}(x)/\partial x_\beta|_{x_2=0} = p_i(x_1) \quad (i = 1, 3), \quad u_2^{(0)}(x)|_{x_2=0} = v_2(x_1) \tag{4.4}$$

5. Let us use the results of solving the above-mentioned problems to analyse the state of stress in the neighbourhood of the vertex of a rectilinear macrocrack of a normal discontinuity that passes over the boundary of rectangular periodicity cells of a composite medium. let there be no shear stresses $\sigma_{12}^{(\varepsilon)}$, $\sigma_{23}^{(\varepsilon)}$ on the edges of this crack and let the normal stress

$$\sigma_{22}^{(\varepsilon)}|_{x_2=\pm 0} = p_2(x_1), \quad |x_1| < a$$

be given. Then, because of the symmetry of the state of stress and strain about the x_1 axis, the consideration can be limited to just the upper half-plane $(x_2 > 0)$ on whose boundary the following conditions should be satisfied (Fig.1):

$$\sigma_{i2}^{(\varepsilon)}|_{x_2=0} = 0 \ (i = 1, 3), \ |x_1| < \infty \tag{5.1}$$

$$\sigma_{22}^{(\varepsilon)}|_{x_2=0} = p_2(x_1), \ |x_1| < a; \ u_2^{(\varepsilon)}|_{x_2=0} = 0, \ |x_1| > a$$

Analysing the state of stress of an inhomogeneous half-plane with the boundary conditions (5.1), several characteristic domains can be extracted in the neighbourhood of the half-plane boundary. We will first consider the domain 4 in the neighbourhood of the points $x_1 = \pm a$, $x_2 = 0$ in the form of a rectangle with the sides $\delta_1 + \delta_2$, δ (Fig.2). This domain contains a finite (fairly small) number of periodicity cells and the state of stress and strain within it should be determined directly from the solution of the elasticity problem without applying asymptotic methods of averaging. The domain 4 is part of an unbounded strip $0 < x_2 < \delta$, $|x_1| < \infty$, within which the sections $0 < x_2 < \delta$, $|x_1| < a$ (domain 1) and $0 < x_2 < \delta$, $|x_1| > a$ (domain 3) are extracted, where utilization of asymptotic solutions of the form elucidated in Sect.2 for domain 1 and the form elucidated in Sect.4 for domain 3 is possible. It must here be noted that the functions $N_{nk\beta}^{(1)}(y)$ and $N_{nk\beta}^{(3)}(y)$ determined from the solutions of the boundary-layer problems (2.18) and (4.2), (4.3) are solutions of boundary-layer type, and consequently, we can set

$$N_{nk\beta}^{(1)}(y) = 0, \ N_{nk\beta}^{(3)}(y) = 0$$

in domain 2 $(x_2 > \delta)$

Therefore, the quantity δ is found by solving problems (2.18) and (4.2) and (4.3) from the condition

$$N^{(1)}_{n k\beta}(y_1, \delta) \approx 0, \; N^{(3)}_{n k\beta}(y_1, \delta) \approx 0$$

The choice of the quantities δ_1, δ_2 (Fig.2) can be made on the basis of the following reasoning. Perturbation of the state of stress due to replacement of the boundary conditions in the nighbourhood of the points $x_1 = a, x_2 = 0$ (crack apices), is localized in domain 4 and does not extend beyond its limits. The characteristic dimensions of the perturbation domain can here be estimated from the known asymptotic forms for the stress in the neighbourhood of a crack. Moreover, it is convenient to select the values of $\delta, \delta_1, \delta_2$ such that the boundaries of domain 4 coincide with the boundaries of the periodicity cells of the composite (see Fig.2).

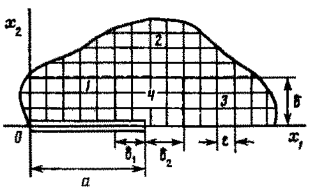

Fig.2

Having the solutions of problems (2.18) and (4.2), (4.3), the stresses can be given on three sides of the rectangle 4, i.e., on the sections $x_1 = a - \delta_1$, $x_1 = a + \delta_2$, $0 < x_2 < \delta$ and $a - \delta_1 < x_1 < a + \delta_2$, $x_2 = \delta$. Since the boundary conditions on the section $a - \delta_1 < x_1 < a + \delta_2$, $x_2 = 0$ are known (relationship (5.1)), the solution of the problem of elasticity theory in domain 4 can be constructed, for instance, using numerical methods.

The formulation of the problem will here to the following

$$\frac{\partial \sigma^{(\varepsilon)}_{\gamma\alpha}(x)}{\partial x_\alpha} = 0, \quad \sigma^{(\varepsilon)}_{\gamma\alpha}(x) = c_{\gamma\alpha\varkappa\beta}(y) \frac{\partial u^{(\varepsilon)}_\varkappa(x)}{\partial x_\beta} \qquad (\alpha, \beta, \gamma, \varkappa = 1,2)$$

$$\sigma^{(\varepsilon)}_{12}(x_1, 0) = 0, \; a - \delta_1 < x_1 < a + \delta_2; \; \sigma^{(\varepsilon)}_{22}(x_1, 0) = p_2(x_1), \; a - \delta_1 < x_1 < a$$

$$u^{(\varepsilon)}_2(x_1, 0) = 0, \; a < x_1 < a + \delta_2$$

$$\sigma^{(\varepsilon)}_{\gamma\alpha}\Big|_{x_1 = a - \delta_1} = \left[c_{\gamma\alpha\varkappa\beta}(y) + c_{\gamma\alpha n\theta}(y)\left(\frac{\partial N_{n\varkappa\beta}}{\partial y_\theta} + \frac{\partial N^{(1)}_{n\varkappa\beta}}{\partial y_\theta}\right)\right]\Bigg|_{y_1=0} \frac{\partial u^{(0)}_\varkappa}{\partial x_\beta}\Bigg|_{x_1 = a - \delta_1}$$

$$\sigma^{(\varepsilon)}_{\gamma\alpha}\Big|_{x_1 = a + \delta_2} = \left[c_{\gamma\alpha\varkappa\beta}(y) + c_{\gamma\alpha n\theta}(y)\left(\frac{\partial N_{n\varkappa\beta}}{\partial y_\theta} + \frac{\partial N^{(3)}_{n\varkappa\beta}}{\partial y_\theta}\right)\right]\Bigg|_{y_1=0} \frac{\partial u^{(0)}_\varkappa}{\partial x_\beta}\Bigg|_{x_1 = a + \delta_2}$$

$$\sigma^{(\varepsilon)}_{\gamma\alpha}\Big|_{x_2 = \delta} = \left[c_{\gamma\alpha\varkappa\beta}(y) + c_{\gamma\alpha n\theta}\frac{\partial N_{n\varkappa\beta}}{\partial y_\theta}\right]\Bigg|_{y_2=0} \frac{\partial u^{(0)}_\varkappa}{\partial x_\beta}\Bigg|_{x_2 = \delta}$$

We note that the constants $h^{(3)}_{2k\beta}$ from the conditions determined by the last equality in (4.3) do not enter into these relationships.

The functions $u^{(0)}_\varkappa(x)$ $(\varkappa = 1, 2)$ are solutions of the following average problem:

$$\langle C_{\gamma\alpha\varkappa\beta}\rangle \, \partial^2 u^{(0)}_\varkappa(x)/\partial x_\alpha \partial x_\beta = 0, \; x_2 > 0$$

$$C^*_{12\varkappa\beta}\partial u^{(0)}_\varkappa(x)/\partial x_\beta\big|_{x_2=0} = 0, \; |x_1| < \infty;$$

$$C^*_{22\varkappa\beta}\partial u^{(0)}_\varkappa(x)/\partial x_\beta\big|_{x_2=0} = p_2(x_1), \; |x_1| < a$$

$$u^{(0)}_2(x)\big|_{x_2=0} = 0, \; |x_1| > a$$

The coefficients in the formulation of the average problem were determined above (see relationships (2.15) and (2.20)).

We note that the possibility of extracting the local domain 4 with known boundary conditions on its contour enables a fairly rigourous analysis to be carried out of the state of stress in this domain for any location of the crack tip in the composite cell, and particularly for the case when the crack apex is on the interfacial boundary of two heterogeneous components.

The proposed method of boundary-layer solutions can also be used in different contact problems of the theory of the elasticity of composite materials of periodic configuration, for instance, in the problem of a stamp.

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